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Assignment V

* Question 1: let $T: V \to V$ be a L.T.
$B = \{b_1, b_2, b_3, b_4\}$ is a basis of V such that:
. $T(b_1) = b_2$, $T(b_2) = b_3$, $T(b_3) = b_4$, $T(b_4) = -b_1 + 2b_3$

i) M_B :

$$M_B = \begin{array}{cccc} T(b_1) & T(b_2) & T(b_3) & T(b_4) \\ \end{array}$$
$$M_B = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

ii) First let us compute $|M_B|$:

$$|M_B| = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 1 \neq 0$$

thus T is invertible $\Rightarrow T^{-1}$ exists.

and, . $T^{-1}(b_2) = b_1$

. $T^{-1}(b_3) = b_2$

. $T^{-1}(b_4) = b_3$

. $b_4 = T^{-1}(-b_1 + 2b_3) \Rightarrow b_4 = -T^{-1}(b_1) + 2T^{-1}(b_3)$

$\Rightarrow T^{-1}(b_1) = 2b_2 - b_4$

iii) Find eigenvalues of T:

First we will get the eigenvalues of M_B.

$C_{M_B}(a) = a^4 - 2a^2 + 1$ (since M_B is a companion matrix)

$= (a^2 - 1)^2$

$= (a-1)^2 (a+1)^2$

thus eigenvalues of M_B are 1 and -1

Hence eigenvalues of T are 1 and -1

. Now let us find the eigenspaces corresponding to the eigenvalues.

$* E_1(M_B)$: $(I_4 - M_B)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 + R_4 \to R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $x_1 = -x_4$, $x_2 = -x_4$, $x_3 = x_4$

and $E_1(M_B) = \{(-x_4, -x_4, x_4, x_4) \mid x_4 \in \mathbb{R}\}$

$= \text{span}\{(-1, -1, 1, 1)\}$

Hence $E_1(T) = \text{span}\{(-b_1 - b_2 + b_3 + b_4)\}$

$\ast$ $E_{-1}(M_B)$: $\quad (-I_4 - M_B)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{-R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{-R_3+R_4 \to R_4} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

thus $x_1 = x_4$, $x_2 = -x_4$, $x_3 = -x_4$

So $E_{-1}(M_B) = \{(x_4, -x_4, -x_4, x_4) \mid x_4 \in \mathbb{R}\}$

$= \text{span}\{(1, -1, -1, 1)\}$

hence $E_{-1}(T) = \text{span}\{(b_1 - b_2 - b_3 + b_4)\}$.

iv) It is obvious that M_B^{-1} is the matrix presentation of T^{-1} with respect to the basis B.

Recall that if λ is an eigenvalue of M_B then $\frac{1}{\lambda}$ is an eigenvalue of M_B^{-1} $(\lambda \neq 0)$

Moreover : $E_\lambda(M_B) = E_{1/\lambda}(M_B^{-1})$

thus eigenvalues of T^{-1} are 1 and -1

and $E_1(T^{-1}) = E_1(T) = \text{span}\{-b_1 - b_2 + b_3 + b_4\}$

$E_{-1}(T^{-1}) = E_{-1}(T) = \text{span}\{b_1 - b_2 - b_3 + b_4\}$

v). We know that $C_T(\alpha) = C_{M_B}(\alpha)$

thus $C_T(\alpha) = (\alpha - 1)^2 (\alpha + 1)^2$

. we can notice that M_B is a companion matrix

thus $m_{M_B}(\alpha) = C_{M_B}(\alpha) = (\alpha-1)^2(\alpha+1)^2$

and $m_T(\alpha) = m_{M_B}(\alpha) = (\alpha-1)^2(\alpha+1)^2$

vi) Since $m_T(\alpha) = (\alpha-1)^2(\alpha+1)^2 \neq (\alpha-1)(\alpha+1)$

we can conclude that T is not diagonalizable.

vii) $M_B^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

thus $C_{M_B^{-1}}(\alpha) = |\alpha I_4 - M_B^{-1}| = \begin{vmatrix} \alpha & -1 & 0 & 0 \\ -2 & \alpha & -1 & 0 \\ 0 & 0 & \alpha & -1 \\ 1 & 0 & 0 & \alpha \end{vmatrix}$

$= \alpha \begin{vmatrix} \alpha & -1 & 0 \\ 0 & \alpha & -1 \\ 0 & 0 & \alpha \end{vmatrix} + 1 \begin{vmatrix} -2 & -1 & 0 \\ 0 & \alpha & -1 \\ 1 & 0 & \alpha \end{vmatrix}$

$= \alpha[\alpha(\alpha^2)] + 1[-2(\alpha^2) + 1(1)]$

$= \alpha^4 - 2\alpha^2 + 1$

thus $C_{T^{-1}}(\alpha) = C_T(\alpha) = \alpha^4 - 2\alpha^2 + 1 = (\alpha-1)^2(\alpha+1)^2$

. we know that T^{-1} is not diagonalizable since T is not diagonalizable thus $m_{T^{-1}}(\alpha) \neq (\alpha-1)(\alpha+1)$

$\longrightarrow$

but the matrix presentation of T^{-1} with respect to some basis is not a companion matrix.

Characteristic polynomial of T^{-1} = Minimum polynomial of T^{-1}. See the notes I WILL post (no points are taken off since there is a missing material that I did not provide)

viii) Let $F: V \longrightarrow V$ s.t $F(v) = -T^4(v) + 2T^2(v)$ $\forall v \in V$

we Know that $C_T(\alpha)|_{\alpha=T} = 0$ - function

thus $T^4 - 2T^2 + I = 0$ where I is the identity map on V.

$\Leftrightarrow -T^4 + 2T^2 - I = 0$

$\Rightarrow \underbrace{-T^4(v) + 2T^2(v)} - I(v) = 0$

$\Rightarrow F(v) - v = 0$

$\Rightarrow F(v) = v$ for every $v \in V$

(ix) Recall that -1 is an eigenvalue of T

then $T(v) = -v$

$T = -I$

$T + I = 0$

$F = 0$ - function

and It is obvious that F thus F^{-1} is doesn't exist.

NOT CORRECT
F is not 0
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be careful here. T(v) = -v only for v in E_{-1}(T) not for all v in V. What you can show here that 0 is an eigenvalue of T + I. Hence Z(T + I) not equal {0}. Thus T + I is not 1-1 and hence T + I is not invertible. To show that 0 is an eigenvalue of T + I. Let v be a nonzero element of E_{-1}(T). Then (T + I)(v) = T(v) + I(v) = -v + v = 0_v.

Note that we conclude that the characteristic polynomial of T + I has 0-constant (note that multiplication of all eigenvalues of T + I (with repetition) = det(Matrix presentation of T + I with respect to some basis of V) = + - constant term of the characteristic polynomial of T + I .

* Question 2:

Let $T: V \longrightarrow V$ such that $IN(V) = 5$

. we know that the degree of the characteristic polynomial is equal to the dimesion of V
dimension

thus $C_T(\alpha)$ has a degree 5.

So $C_T(\alpha)$ ~~has~~ is a polynomial of odd degree and we know that every polynomial of odd degree ~~has~~ must have at least one real root.

Thus T must have at least one real eigenvalue say α. and a corresponding eigen function $v \in V$, $v \neq 0$,
eigenvector or eigen-element

such that $T(v) = \alpha v$

* Question 3:

let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$

since A is a companion matrix

then $C_A(\alpha) = m_A(\alpha) = \alpha^3 - 3\alpha + 2$

$= (\alpha - 1)^2 (\alpha + 2)$

and since $m_A(\alpha) \neq (\alpha - 1)(\alpha + 2)$ thus A is

not diagonalizable.

therefore A is 3×3 matrix, such that $C_A(\alpha) = m_A(\alpha)$

and A is not diagonalizable.

* Question 4:

$$\det A = \begin{pmatrix} 0 & 0 & 6 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{pmatrix}$$

since A is a companion matrix we have:

$$C_A(\alpha) = m_A(\alpha) = \alpha^3 - 6\alpha^2 + 11\alpha - 6$$
$$= (\alpha-1)(\alpha-2)(\alpha-3)$$

and A is diagonalizable. ~~since~~

thus A is a 3x3 matrix, such that $C_A(\alpha) = m_A(\alpha)$

and A is diagonalizable.